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Anomalous multifractality of conductance jumps in a hierarchical percolation model

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Abstract. We have investigated the multifractal scaling of conductance jumps in a hierarchical percolation lattice, resulting from cutting the current carrying bonds in the lattice. Due to the iterative nature of the model, exact renormalization group (RG) equations are obtained and used to extract the minimum conductance jump of the lattice. We find an asymptotically analytic expression for the minimum conductance jump,

$$\Delta g_{\min} \simeq \exp[-c(\log L)^2]$$

decreasing faster than any power law. We observed slow convergence to the asymptotic behaviour due to the importance of the irrelevant terms in the RG equations at low generations of the lattice. Numerical calculations are performed in order to validate the analytic results and to calculate the $f-\alpha$ spectrum to confirm left-sided multifractality as proposed by Lee and Stanley.

1. Introduction

Recently, there has been an increased interest in the critical behaviour of random resistor networks. In particular, attention has been paid to the breakdown of multifractality in a range of negative moments [1]. Such anomalous multifractal measure was originally found in studies of diffusion limited aggregations (DLAs) by Schwarzer *et al* [2]. At the bottom fjord in DLA structures, it is found that the minimum growth probability decays exponentially with size L; the free energy $\tau(q)$ is singular at $q=0$ and fails to be defined for all $q<0$. These are anomalous measures that Lee and Stanley [3] originally discovered and Mandelbrot *et al* [4] named 'left-sided multifractality'. It is surprising to find exact renormalizability or self-similarity will result in failure of scaling of the partition function.

Very recently, Nagatani [5] extended a hierarchical percolation model originally proposed by de Arcangelis *et al* [6] to model the electrical properties of the percolation backbone of a random resistor network. The lattice contains rare clusters which take into account the faster decreasing minimum current fraction than a power law. The anomalous multifractal measure of the current distribution was studied analytically [5]. In this paper we adopt the $b=3$ version of the hierarchical model (as shown in figure 1) and calculate numerically conductance jump (or resistance jump) distribution, resulting from cutting the current-carrying bonds of the lattice [7]. We obtain the multifractal spectrum for conductance jump.

The paper is organized as follows. In section 2 we describe the construction of the hierarchical lattice. In section 3 we use an exact renormalization group (RG) method

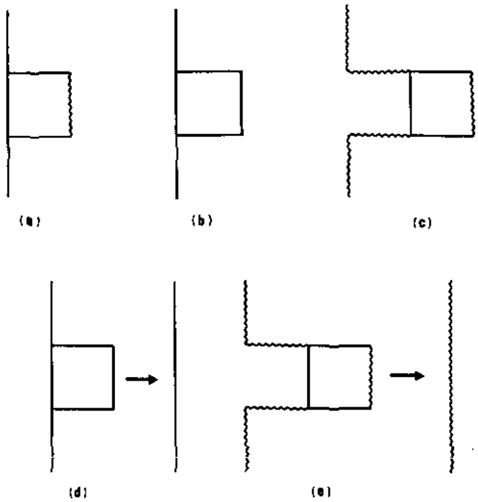

Figure 1. The construction of the hierarchical lattice: (a) initiator, (b) regular cluster and (c) rare cluster. The renormalization process: (d) regular cluster → regular bond, and (e) rare cluster → rare bond.

to obtain the minimum conductance jump and derive an asymptotically analytic expression for it. We show that the result can be understood with Tellegen's theorem. In section 4 we describe a simple algorithm which allows us to find the complete set of values of conductance jump at any arbitrary generation. Section 5 deals with two important ways to characterize multifractality. The method of Lee and Stanley [3] will be used to study the finite-size multifractal spectrum in this anomalous case. In section 6, the relationship between the current distribution and the conductance jump distribution will be established through Tellegen's theorem. In the final section we present some concluding remarks.

2. Hierarchical percolation model

The hierarchical lattice is generated by replacing each bond by the corresponding generator iteratively. Starting from the initiator (figure 1(a)), the solid bond is replaced by the regular cluster (figure 1(b)), while the wavy bond is replaced by the rare cluster (figure 1(c)) to obtain the first generation. The lateral size is therefore increased by a factor of three. The second generation is obtained from the first generation by replacing each bond with each generator. The process is repeated *ad infinitum* to obtain the deterministic model.

The fractal dimension of the hierarchical lattice can be determined as follows. Let C_n, G_n and F_n denote, respectively, the number of bonds in the initiator, and the regular and rare clusters at the nth generation. By construction, each time when we go from the kth generation to the next generation we increase the lattice size by a factor of three while we increase G_k by a factor of six. That is,

$$G_{k+1} = 6G_k \qquad k \geqslant 0,\ G_0 = 1.$$

We find $G_n = 6^n$. From figure 1(c),

$$F_{k+1} = 5F_k + 3G_k \qquad k \geqslant 0,\ F_0 = G_0 = 1.$$

We obtain $F_n = 3(6^n) - 2(5^n)$ and $\lim_{n\to\infty} F_n/G_n = 3$. From figure 1(*a*),

$$C_{k+1} = 5G_k + F_k \qquad k \geqslant 1,\ C_1 = 6.$$

We obtain $C_n = 6^n + 2(6^{n-1} - 5^{n-1})$ and $\lim_{n\to\infty} C_n/G_n = \frac{4}{3}$. One thus finds that $d_b = \lim_{n\to\infty} \log_3(C_{n+1}/C_n) = \lim_{n\to\infty} \log_3(G_{n+1}/G_n) = \log 6/\log 3 \approx 1.63$. We can also determine the cut bond dimension $d_c = \log 2/\log 3 \approx 0.63$ and the conductivity exponent $t/\nu \approx (\log 11 - \log 4)/\log 3 = 0.92$ (see below). These values are close to the known percolation values: $d_b = 1.62$, $d_c = 0.75$, $t/\nu = 0.97$ [8].

3. Minimum conductance jump

Due to the iterative nature of the lattice, one can find exact RG equations for the regular and the rare clusters. Figure 1(*d*) shows the renormalization for the regular cluster. Let g and f be the conductances for the regular and rare bonds at the kth generation. We find the transformed values g' and f' at the $(k+1)$th generation,

$$g' = \tfrac{1}{2}g \oplus [g + (\tfrac{1}{2}g \oplus g)] = \tfrac{4}{11}g \tag{1}$$

where $g_1 \oplus g_2 = g_1 g_2/(g_1 + g_2)$ denotes the series combination of two conductances g_1 and g_2. One thus finds that $t/\nu = \log_3(\frac{11}{4}) \approx 0.92$ in good agreement with the known result for percolation. Also, figure 1(*e*) shows the renormalization of the rare cluster. We have

$$f' = \tfrac{1}{4}f \oplus [g + (\tfrac{1}{2}g \oplus f)].$$

Let $x = f/g$ and $x' = f'/g'$, we find

$$\frac{1}{x'} = \frac{4}{11}\left(\frac{4}{x} + \frac{1+2x}{1+3x}\right). \tag{2a}$$

As $n \to \infty$, $x' \to 0$, we obtain $x' = \frac{11}{16}x$. With $x_0 = 1$, we find

$$x_n = (\tfrac{11}{16})^n. \tag{2b}$$

In figure 2, we show the conductance g for the regular cluster and the conductance f for the rare cluster as a function of L on a log-log plot. One can see that the rare

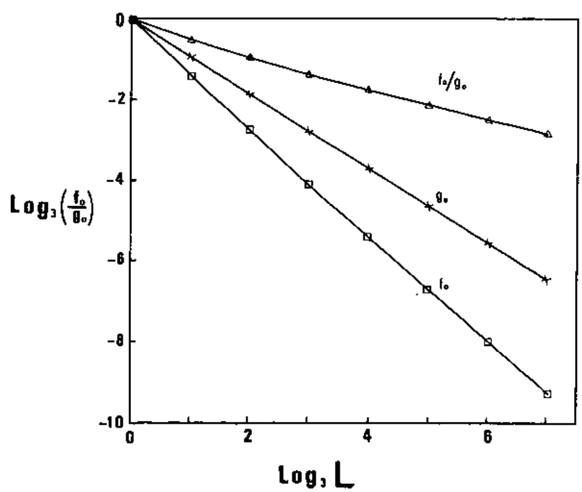

Figure 2. Log-log plot of rare/regular conductance ratio.

conductance f decreases faster than the regular conductance g as the size of the lattice increases (both exhibit power law decay). In the same figure we also plot the ratio $x=f/g$ as a function of L. The figure confirms the asymptotic relation $(2b)$. However, the convergence to asymptotic behaviour is quite slow due to the irrelevant term which prevails in equation $(2a)$.

In order to calculate the minimum conductance jump $\Delta g_{\min}$ which occurs upon cutting the rare bond inside the plaquette, let $d=zg$ be the conductance of the defect rare bond, then

$$d'=\tfrac{1}{4}f\oplus[g+(\tfrac{1}{2}g\oplus d)]$$

or

$$\frac{g'}{d'}=\frac{4}{11}\left(\frac{4}{x}+\frac{1+2z}{1+3z}\right). \tag{3}$$

Let us take the logarithm of equation (3), differentiate it with respect to z and let $z=x\to 0$, then

$$\left.\frac{\Delta d'}{d'}\right|_{z=x}=\frac{\Delta g'/g'}{f'/g'}=\frac{1}{4}x\Delta z=\frac{1}{4}x\left(\frac{\Delta g}{g}\right)$$

i.e.

$$\left(\frac{\Delta g_{\min}}{g}\right)_k=\frac{1}{4}x_k x_{k-1}\left(\frac{\Delta g_{\min}}{g}\right)_{k-1}. \tag{4}$$

Iterating the relation n times with the use of equation $(2b)$, one finds

$$\left(\frac{\Delta g_{\min}}{g}\right)_n=\left(\frac{1}{4}\right)^n\left(\frac{11}{16}\right)^n\left(\frac{11}{16}\right)^{2(n-1)}\cdots\left(\frac{11}{16}\right)^2\left(\frac{\Delta g_{\min}}{g}\right)_0.$$

Taking the logarithm, one arrives at

$$\log_3\left(\frac{\Delta g_{\min}}{g}\right)_n=-\left[2n\log_3 2+n^2\log_3\left(\frac{16}{11}\right)\right]+\log_3\left(\frac{\Delta g_{\min}}{g}\right)_0. \tag{5}$$

Thus, the asymptotic behaviour of $\Delta g_{\min}$ is dominated by the n^2 term, or

$$\Delta g_{\min}\approx\exp[-c(\log L)^2]. \tag{6}$$

In figure 3, we present a plot for the logarithm of minimum conductance jump against several possible forms of $F(k)$, for $1\leqslant k\leqslant 7$, where $k=\log_3 L$. First we want to check if $\Delta g_{\min}$ follows a power law. The first form is $F(k)=k$. We obtain a curve bending downwards towards large k, indicating that $\Delta g_{\min}$ decreases faster than a power law. We then try the asymptotic form of equation (5): $F(k)=2k\log_3 2+k^2\log_3\frac{16}{11}$. The small-$k$ region does not fit very well, indicating that the convergence to the asymptotic behaviour is indeed slow. Alternatively, we try the form $F(k)=0.1(k+5)^2$. Here an excellent fit is found. It is probably due to the presence of irrelevant terms that leads to power law corrections to the asymptotic behaviour. Thus, we find that the minimum conductance jump does not scale linearly with L but instead follows the asymptotic behaviour of equation (6).

We may understand this behaviour with Tellegen's theorem [9]. For a certain external current, by Tellegen's theorem,

$$\Delta R=\Delta g/g^2=\delta R I_{\min}^2/I_0^2$$

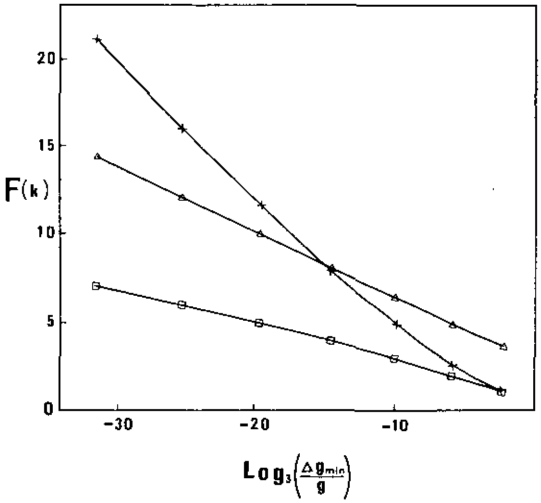

Figure 3. Plot of logarithm of minimum conductance jump against lateral size. Several forms of $F(k)$, with $k = \log_3 L$ is proposed (see text).

where $I_0, \Delta R$ are, respectively, the external current and resistance jump of the entire lattice, $I_{\min}$ and δR being the current and resistance jump relevant to the innermost plaquette of the lattice. If we fix the external current $I_0 \approx 1$ and for $\delta R \approx 1$, we find that $\Delta g_{\min} \approx g^2 I_{\min}^2$. Using the minimum current $I_{\min} \approx \exp[-(c/2)(\log L)^2]$ from [5], we recover the desired result (equation (6)).

4. The complete set of conductance jump values

In order to extract the multifractal spectrum of conductance jumps (Δg) and to confirm the above asymptotic results for $\Delta g_{\min}$, here we would like to find all possible values of Δg by cutting the current-carrying bonds in the lattice, one at a time. The evaluation of all possible Δg values is not straightforward and we give a brief description in the following.

Due to the iterative nature of the lattice, we may accomplish this recursively. As shown in figure 1(b), the regular cluster does not contain any rare bonds in it and its renormalization can be performed independently. Starting with the first generation, each bond has a unit conductance. When a bond is to be removed from the cluster, there are three distinct places that it can be done: the series position with two-fold multiplicity, the short parallel position with only one case and the long parallel position with three-fold multiplicity. The conductance of the defect cluster (with one single bond being removed) can be evaluated for each of the three distinct cases. For convenience, we record the conductance value d_i together with the multiplicity m_i in a data file and $\Sigma\, m_i = G_1 = 6$ gives a check of the correctness.

For the second generation, we take the regular cluster but with each bond filled with conductance g_1. We take the defect bond from the first generation to form the defect cluster and evaluate the conductance for each of the three distinct cases. Again we obtain the multiplicity together with the conductance value; $\Sigma\, m_i = G_2$ gives a check of the correctness of the calculations. We have evaluated the conductance of the regular cluster up to the seventh generation.

For the rare cluster as shown in figure 1(c), both regular and rare bonds are contained. Starting with the first generation, it contains four rare bonds in series and one rare bond in parallel inside the plaquette; there are also three regular bonds, one at the short parallel position and the other two at the long parallel position. We record the conductance value as well as the multiplicity. Again $\Sigma\, m_i = F_1 = 8$. For the second generation, we use the defect bonds for the regular cluster as well as for the rare clusters from the first generation and repeat similar calculations. We check $\Sigma\, m_i = F_2 = 58$. We have calculated the conductance of the rare cluster up to the seventh generation.

We then combine both the regular bonds and the rare bonds to form our hierarchical lattice. The conductance and the multiplicity associated with it are evaluated and recorded. This gives us a complete list of conductance jump values for our further analysis. With a close examination of the Δg data, we find that the maximum jumps occur on cutting the outmost series bonds, where $\Delta g_{\max}$ is equal to the full conductance at that generation while $\Delta g_{\min}$ occurs well inside the plaquette. The jump values spread out a wide range over several orders of magnitude.

5. Multifractal spectrum of conductance jumps

There are several ways to characterize multifractality. The most precise way is to study moments of the distribution of conductance jumps:

$$M_q = \sum_i m_i (\Delta g_i(L))^q \tag{7}$$

where q is the moment order, i being the label of the ΔG values. For $q \geqslant 0$ and sufficiently large L, these moments scale with size L as

$$M_q \sim L^{-\tau(q)}. \tag{8}$$

A multifractal distribution will be such that $\tau(q)$ is not an affine function of q (i.e. absence of constant gap scaling).

The large conductance jump of the regular cluster gives dominant contributions to the moment for $q \geqslant 0$. The fixed support of $\alpha_{\min} = \log_3 \Delta g_{\max}$ is as expected for $d_c \approx 0.63$. On the other hand, for $q < 0$, the minimum conductance jump $\Delta g_{\min}$ dominates the negative moments of equation (7). Therefore, according to the exponential asymptotic behaviour of equation (6), the negative moments cannot scale.

In figure 4(a), we plot $\log_3 M_q$ against $k = \log_3 L$ for $0 \leqslant q \leqslant 5$ and we find a linear relationship for $k \geqslant 2$. This confirms the rapid convergence to the scaling of equation (8), though not being characterized by a single gap exponent. $\tau(q)$ is given by the negative of the slope of these plots. The slope $-\tau(q=0)$ for $q = 0$ is simply the fractal dimension of the hierarchical lattice; we find $d_b \approx 1.63$. In figure 4(b), we plot $\log_3 M_q$ against $k = \log_3 L$ for $-5 \leqslant q \leqslant 0$. In addition, figure 5 shows $\tau(q)$ as a function of q. We see that there is again remarkably rapid convergence as a function of L for $q \geqslant 0$. On the other hand, for $q < 0$, there appears to be no convergence at all and $\tau(q)$ is therefore not defined for $q < 0$, in contrast to the well-defined $\tau(q)$ for the $q \geqslant 0$ case.

Here we follow Lee and Stanley [3] to calculate the finite-size $f(\alpha)$ spectrum to exhibit the behaviour of left-sided multifractality. Let us define a parameter ε associated with the conductance jump values:

$$\varepsilon = -\log \Delta g_i(L)/\log L. \tag{9}$$

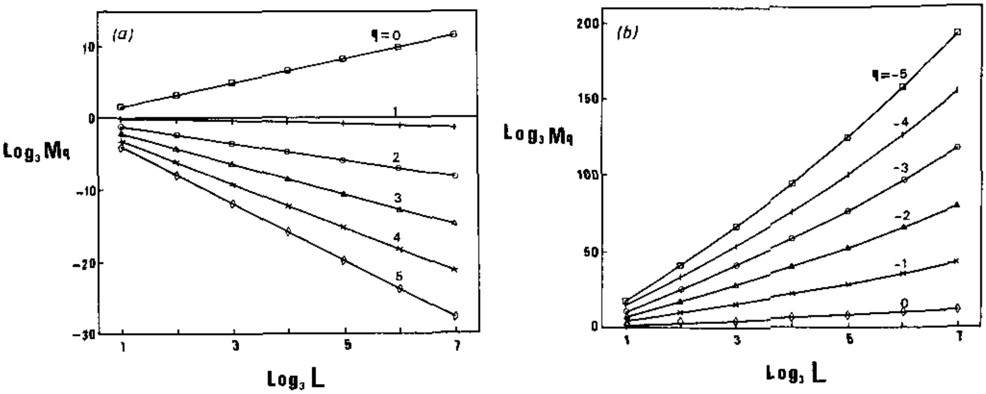

Figure 4. (*a*) Log–log plot of positive moments against L showing scaling behaviour (though non-constant gap). (*b*) Same as (*a*), negative moments against L showing no scaling behaviour.

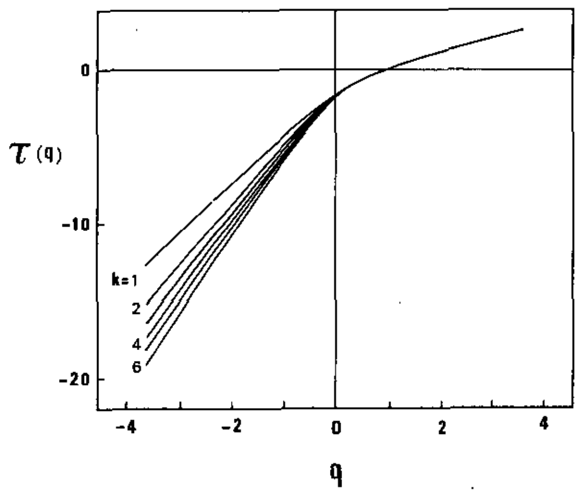

Figure 5. Finite-size $\tau(q)$ behaviour, $1 \leqslant k \leqslant 6$, showing no convergence to scaling for $q<0$.

In analogy to thermodynamics, we define a partition function

$$M_q(L)=\sum_{\varepsilon} D(\varepsilon, L)L^{-q\varepsilon} \tag{10}$$

where $D(\varepsilon, L)$ is the density of states function, identical to the multiplicity associated with the conductance jump value. As we study the finite-size behaviour we define the free energy $\tau(q)$ in the limit of infinite size:

$$\tau(q)=\lim_{L\to\infty} \tau(q, L) \tag{11}$$

where $\tau(q, L)=-\log M_q(L)/\log L$. With the Legendre transformation of $\tau(q, L)$, we obtain the finite size $f(\alpha)$ spectrum

$$f(q, L)=q\alpha(q, L)-\tau(q, L) \tag{12}$$

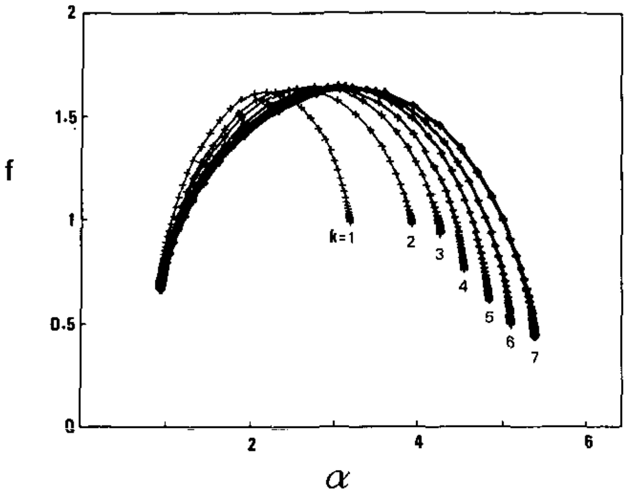

Figure 6. Finite-size $f(\alpha)$ behaviour, $1 \leq k \leq 7$ showing left-sided multifractality.

where $\alpha(q, L) = \partial\tau(q, L)/\partial q$ is the variable conjugate to q. In figure 6, we plot the finite-size $f(\alpha)$ spectrum of conductance jumps. The right-hand side that corresponds to small conductance jumps does not scale or converge. It exhibits a similar behaviour to the multifractal spectrum of DLA, namely that the right-hand side does not scale, or $\alpha_{\max}$ increases with L and hence the partition function does not scale as a power law and $f(\alpha)$ is a straight line with slope q at the large-α region.

6. Relationship between current distribution and conductance jump distribution

Here we should mention that the distribution of conductance jumps is related to that of the resistance jump as well as the current distribution [7]. Cutting bonds carrying large currents in the lattice will result in large conductance jumps or resistance jumps. As a consequence, we found that these spectra are related to each other.

Let us first consider the spectra of Δg and ΔR. For small Δg, $\Delta g/g \ll 1$, $\Delta R \sim \Delta g/g^2$. Suppose $g \sim L^{-t/\nu}$, $\Sigma\, \Delta g^q \sim L^{-\tau(q)}$ and $\Sigma\, \Delta R^q \sim L^{-\rho(q)}$. Thus, $\rho(q) = \tau(q) - 2qt/\nu$, $\alpha_R = \alpha_g - 2t/\nu$ and $f_R(\alpha_R) = f_g(\alpha_g)$. We find that the conductance jump and the resistance jump spectra are equal except for a shift of a value $2t/\nu \approx 1.84$ of the α_g axis. However, we find that the hierarchical lattice contains plenty of one-dimensional links, which would violate the approximation $\Delta R \sim \Delta g/g^2$, as the condition $\Delta g \ll g$ no longer holds. Therefore, the $q > 0$ part of the spectra is not simply related to a simple shift of the α-axis. Since cutting one-dimensional links makes ΔR infinite, the $f_R(\alpha_R)$ spectrum will be truncated.

We next consider the spectra of the resistance and the current distributions. As the condition $\delta R_1 \ll R_1$ holds for large positive α_R, we can use Tellegen's theorem [9] $\Delta R_L = \delta R_1 (I_1/I_L)^2$, where I_L and ΔR_L are, respectively, the external current and resistance jump of the entire lattice, I_1 and δR_1 being the current and resistance jump relevant to the mesh size of the lattice. If we fix the external current I_L, for certain δR_1, $\Delta R_L \propto I_1^2$. If $\Sigma\, I_1^q \sim L^{-\sigma(q)}$, then $\rho(q) = \sigma(2q)$, $\alpha_R(q) = 2\alpha_I(2q)$, and $f_R(\alpha_R(q)) = f_I(\alpha_I(2q))$. Since $f_R(\alpha_R)$ is defined only for $q < 0$, the $f_I(\alpha_I)$ spectrum for $q < 0$ can be obtained by rescaling the α_R axis by a factor of two.

7. Conclusion

In conclusion, we have investigated the multifractal scaling of conductance jumps in a hierarchical percolation lattice, resulting from cutting the current-carrying bonds in the lattice. Due to the iterative nature of the lattice, exact RG equations are obtained and used to extract the minimum conductance jump of the lattice. We find an asymptotic analytic expression for the minimum conductance jump,

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decreasing faster than any power law. We observed slow convergence to the asymptotic behaviour due to importance of the irrelevant terms in the RG equations at low generations of the lattice. Numerical calculations are performed in order to validate the analytic results and to calculate the $f-\alpha$ spectrum to confirm the left-sided multifractality proposed by Lee, Stanley and coworkers [3].

Acknowledgments

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